Measures of Tipping Points, Robustness, and Path Dependence

Aaron L Bramson

Motivation

- The idea of tipping points is important for topics as diverse as segregation, marketing, rioting, global warming, idea diffusion, election outcome, and many, many more.

- Robustness considerations have extended beyond engineering and ecology to political regimes, computer algorithms, and decision procedures.

- Analyzing and exploiting path dependence plays a significant role in technology spread, institutional design, legal theory and the evolution of culture.

- However these concepts have not been generally and formally defined and, as a result, the terms' uses across these various applications are hardly consistent.

Motivation

- Complex systems analyses need to capture processes rather than a series of static snapshots.
- Existing statistical techniques are ill-suited to measure properties of system dynamics.
- Developing techniques to measure these features could provide a means to compare models across disciplines.
- Provides a conceptual benefit for understanding and categorizing system behaviors.
- Makes the inventor very popular at social gatherings.

Building a Markov Model

- Markov models are comprised of a set of states and the probabilistically weighted transitions among those states.

- The Markov model representation must be built in a specific way to run the properties of system dynamics analysis.

    - A state in the Markov model is a complete specification of the Q aspects of one configuration of the system.

$$S_i = \{X_{1(i)}, X_{2(i)}, \ldots X_{Q(i)}\}$$

    - Example: If our system is an iterated game played by six players each with four possible actions then each state of the system has six aspects and each aspect takes on one of four values. That is $S_i = \{a(P1_{(i)}), a(P2_{(i)}), \ldots, a(P6_{(i)})\}$ and a particular state S_3 might be $\{a_3, a_2, a_3, a_1, a_4, a_3\}$.

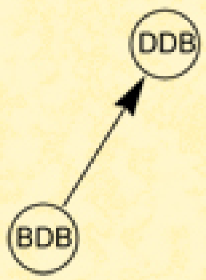
Building a Markov Model

- For each independent trial start with the initial state.

BDB

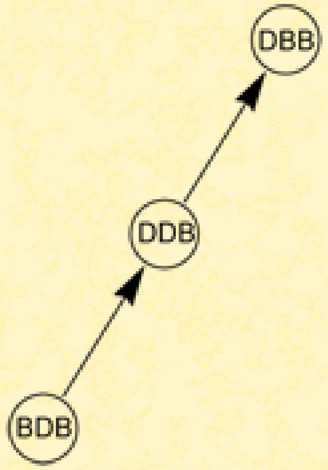
Building a Markov Model

- Record which state it transitions into.



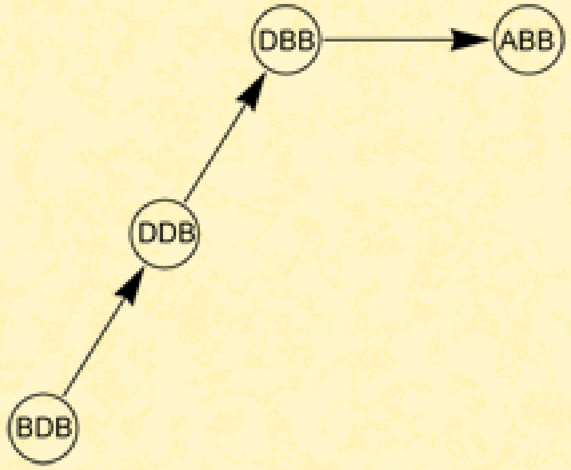
Building a Markov Model

- And do it again...



Building a Markov Model

- ...and again...



Building a Markov Model

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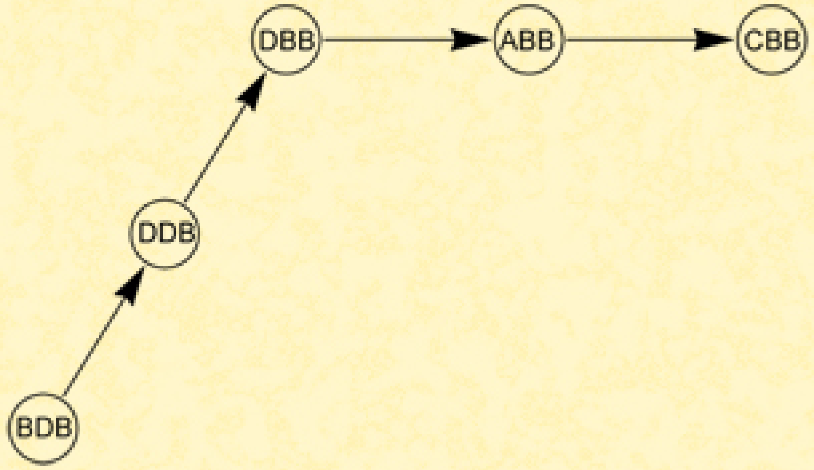
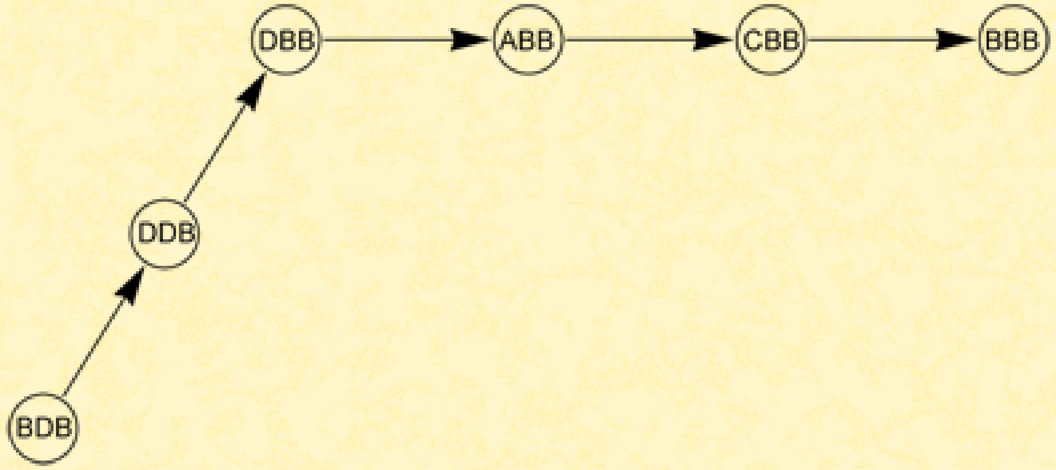


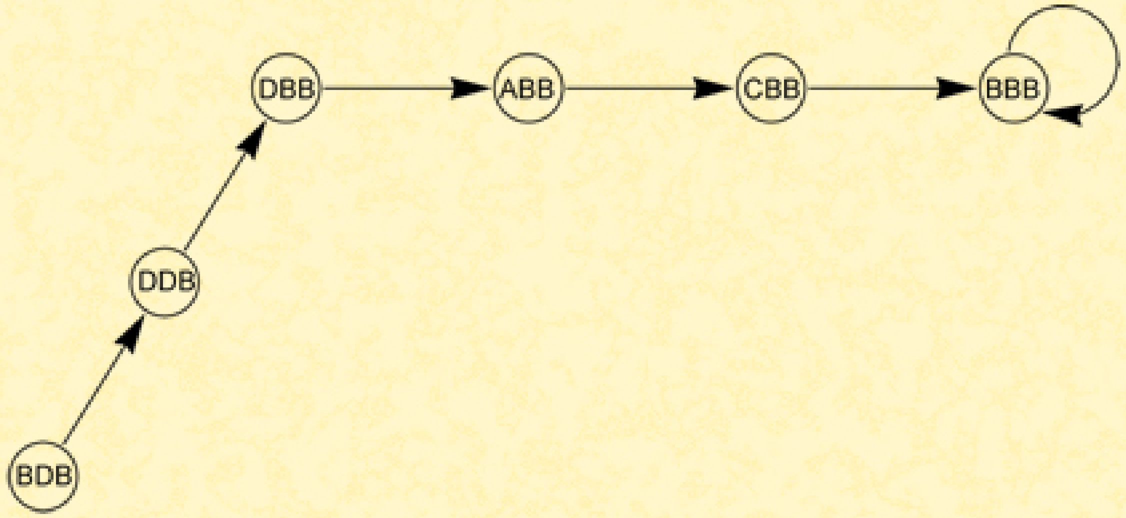
Building a Markov Model

- ...and again...



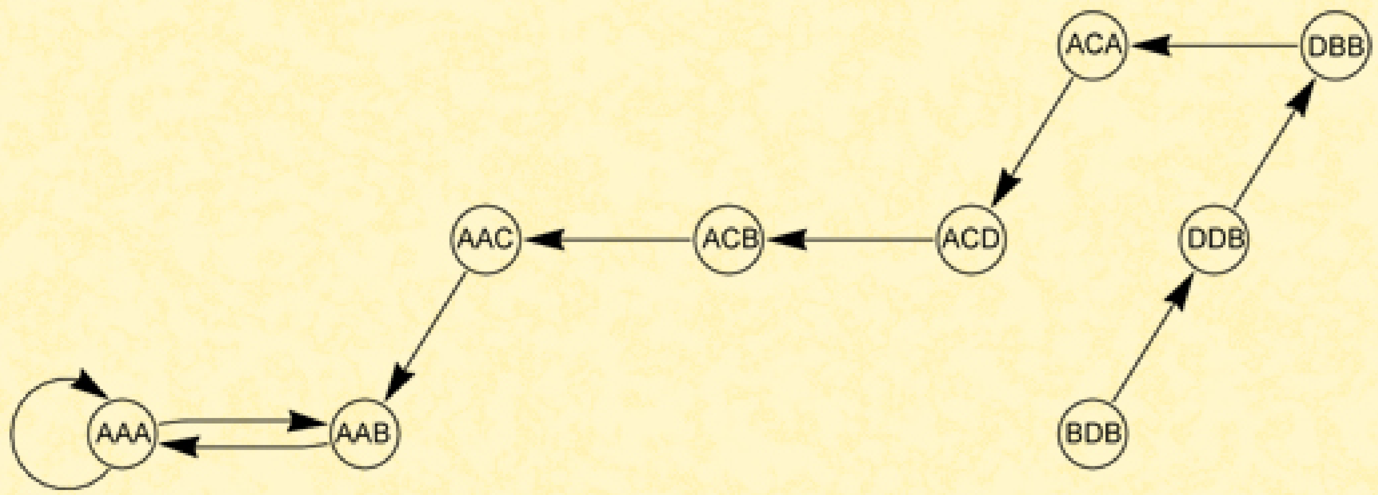
Building a Markov Model

- ...until you reach what looks to be an equilibrium or you've run out of data for that trial.



Building a Markov Model

- Then run another independent trial and track its dynamics through the state space.



Building a Markov Model

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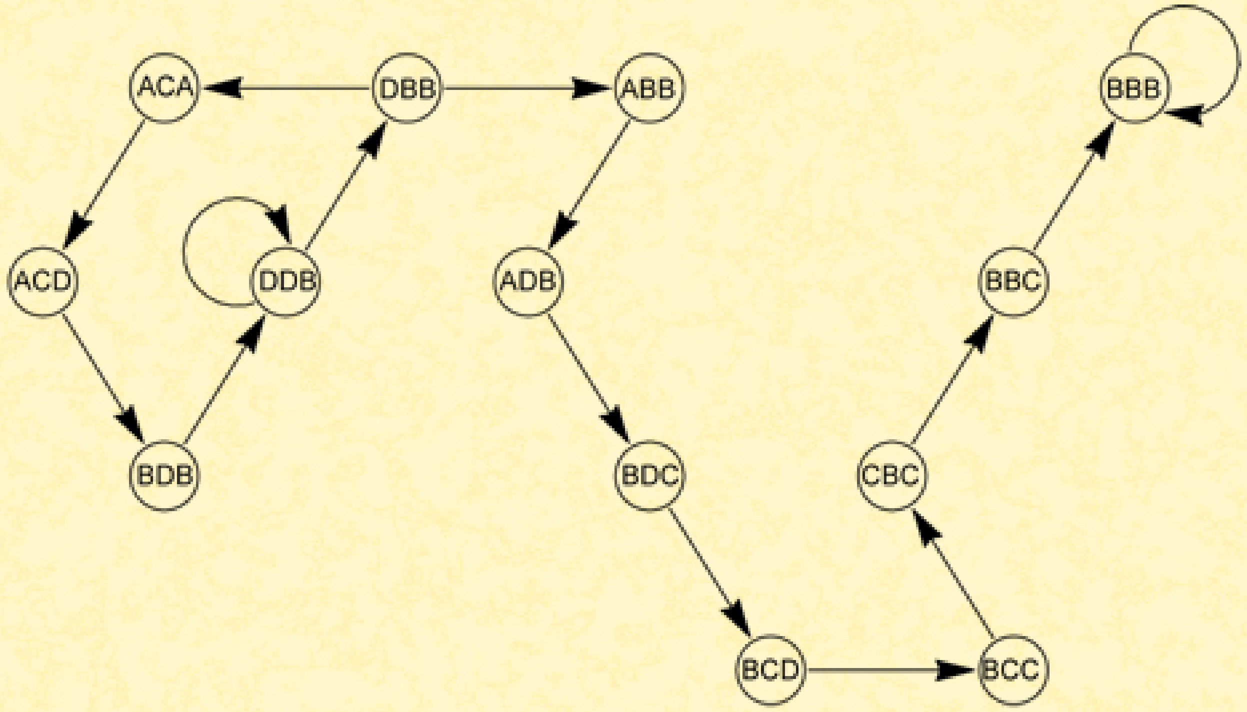
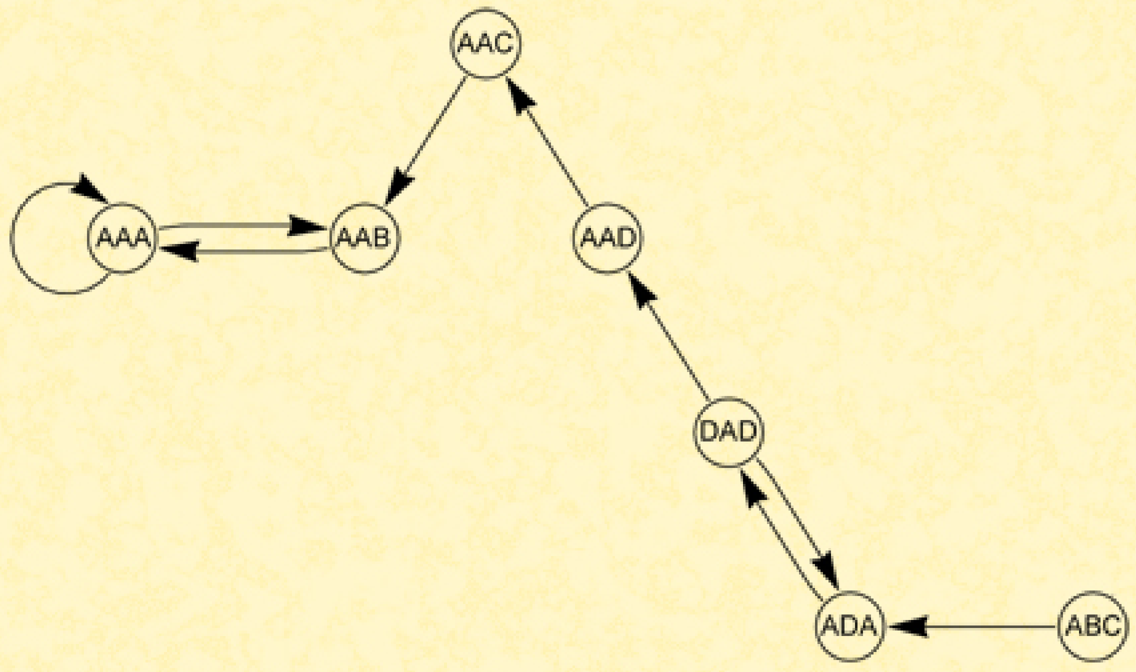


Building a Markov Model

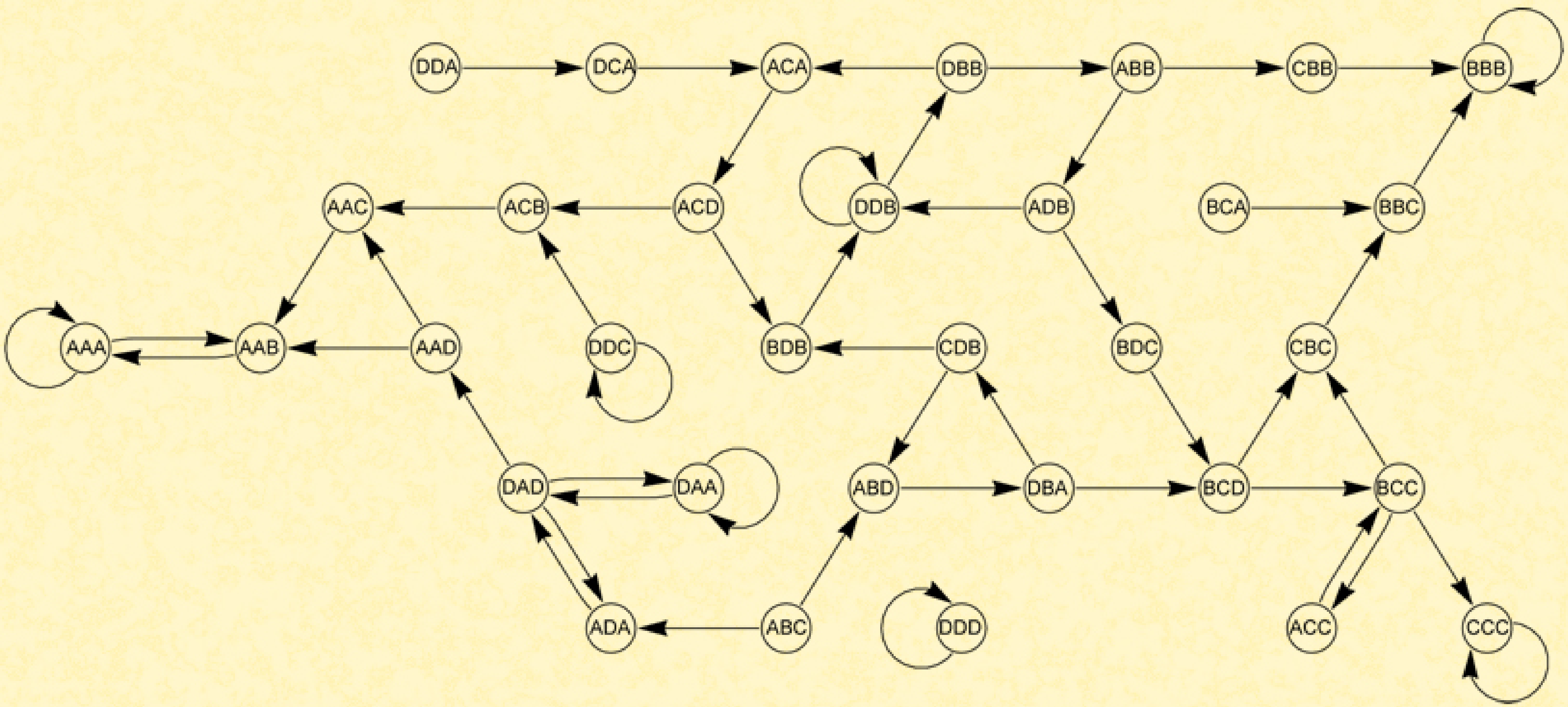
- Then run yet another independent trial and track its dynamics through the state space.

- Continue to do this as long as is reasonable.



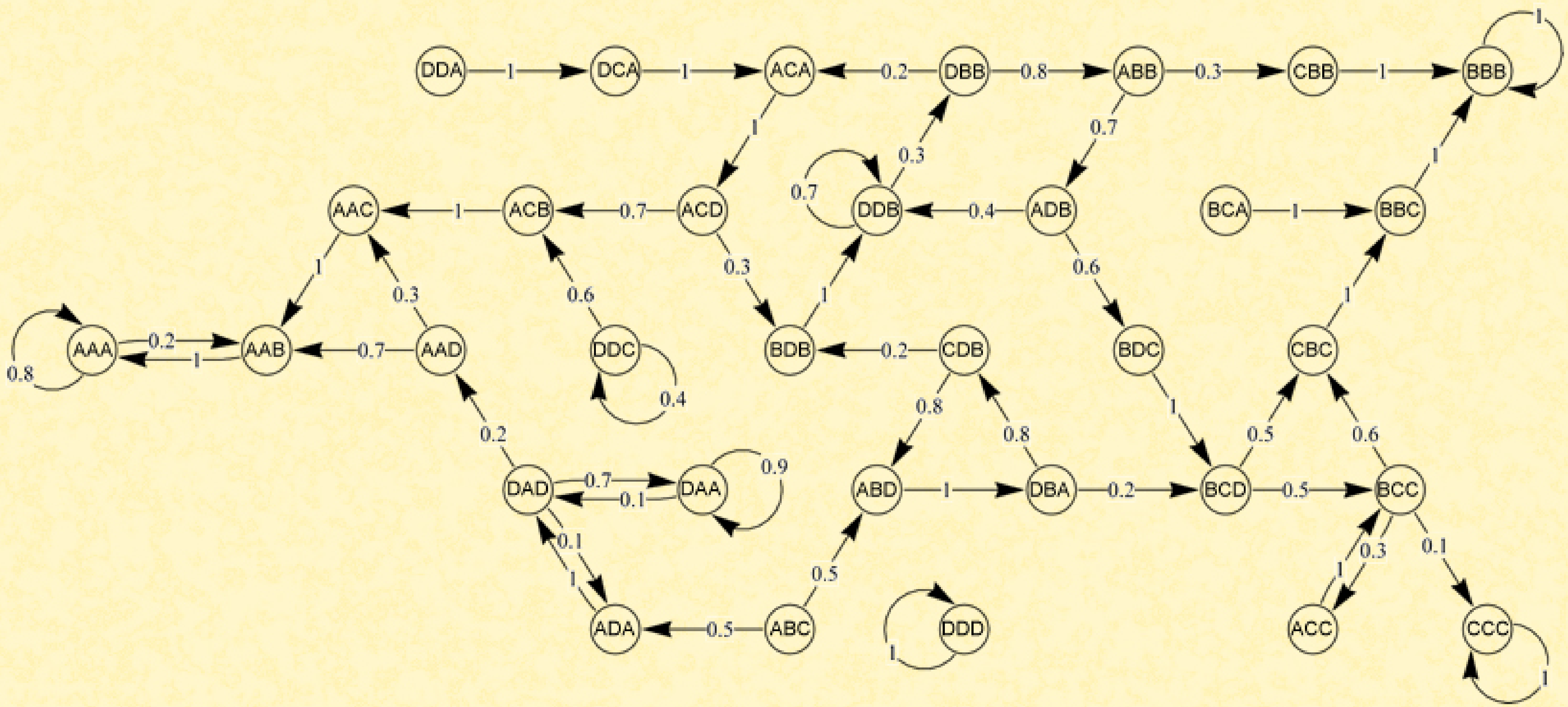
Building a Markov Model

- And when you've run through all your data or have done as many simulations as you can stand, compile them into one diagram that captures all the observed dynamics.



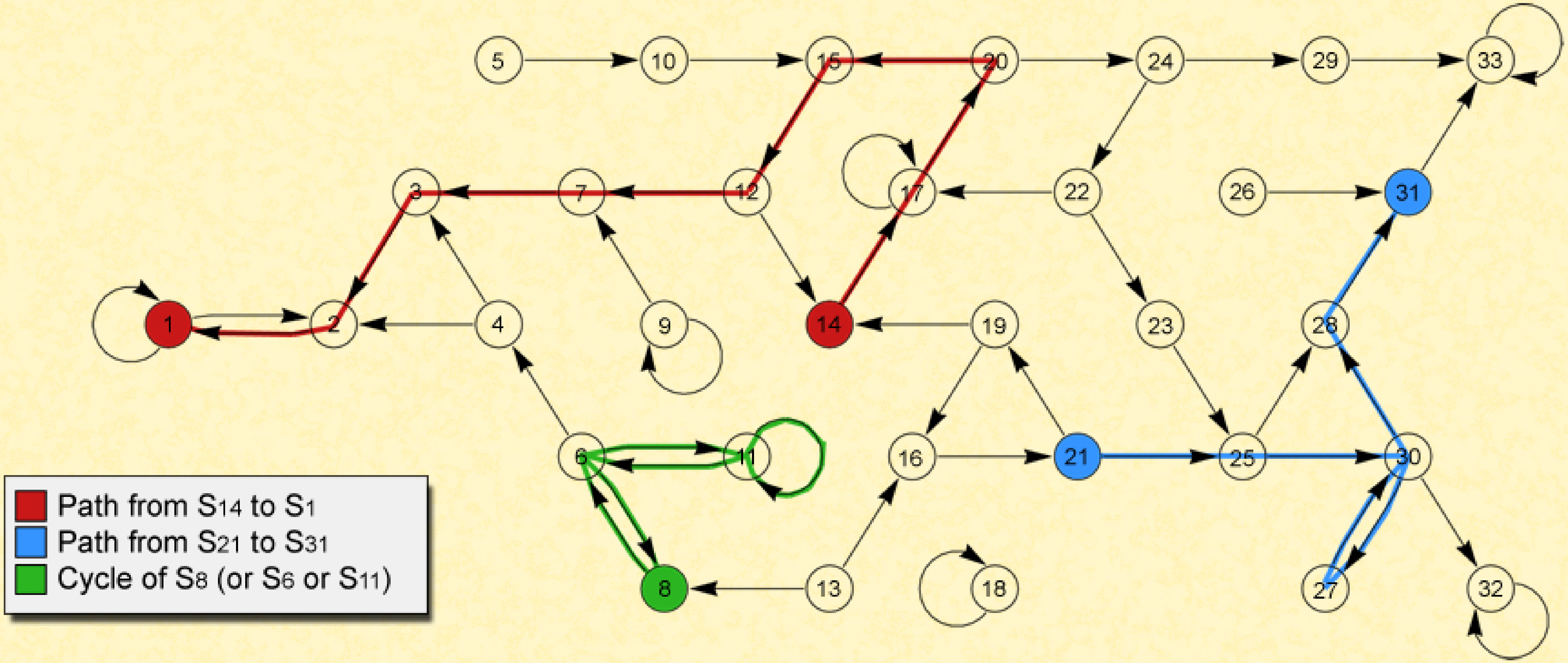
Building a Markov Model

- And that's not all!!! By using the frequency of transitions from your data you can produce the desired Markov model representation of your system's dynamics.



Paths and Cycles

- A path is an ordered collection of states and transitions such that from each state there exists a positive probability to transition to the successor state within the collection. A path from S_i to S_j denoted $\tilde{S}(S_i, S_j)$ is the set of states **S** such that
  (i) $s_0 = S_i$ in **S**
  (ii) There exists T such that for all $t < T$ $P(s_{t+1} \text{ in } \mathbf{S} \mid s_t \text{ in } \mathbf{S}) > 0$
  (iii) $s_T = S_j$ in **S**

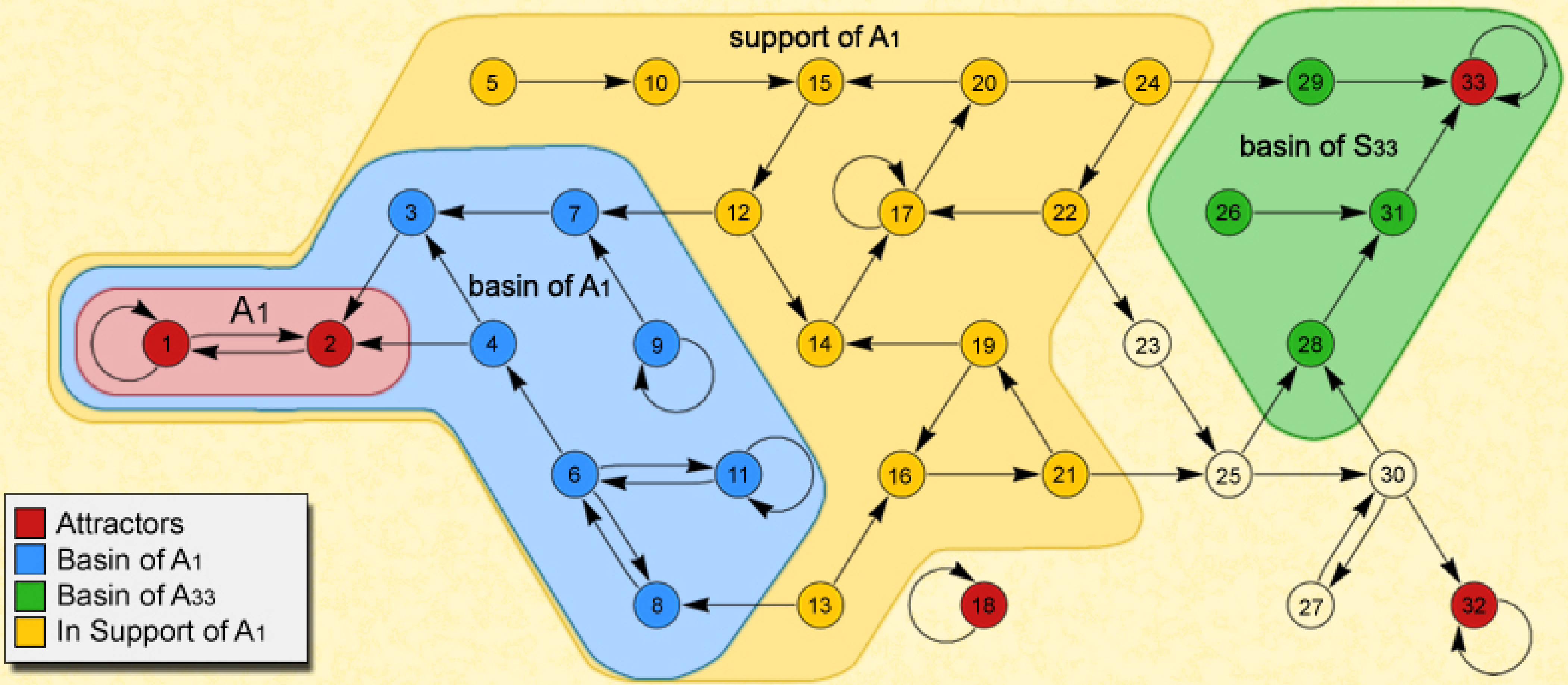


Landmarks in System Dynamics

- A system state that always transitions to itself is called an equilibrium or stable state. An equilibrium e_i is a state S_i such that $P(s_{t+1} = S_i | s_t = S_i) = 1$.

- Those states from which the system will eventually move into a specific attractor are said to be in that attractor's basin of attraction. The basin of A_i or $\mathbf{B}(A_i)$ is a set of states $\mathbf{S}$ such that there exists an $h \geq 0$ $P(s_{t+h} = A_i | s_t \text{ in } \mathbf{S}) = 1$.

- The support of a state (also known as its in-component) is the set of states which have a path to it.

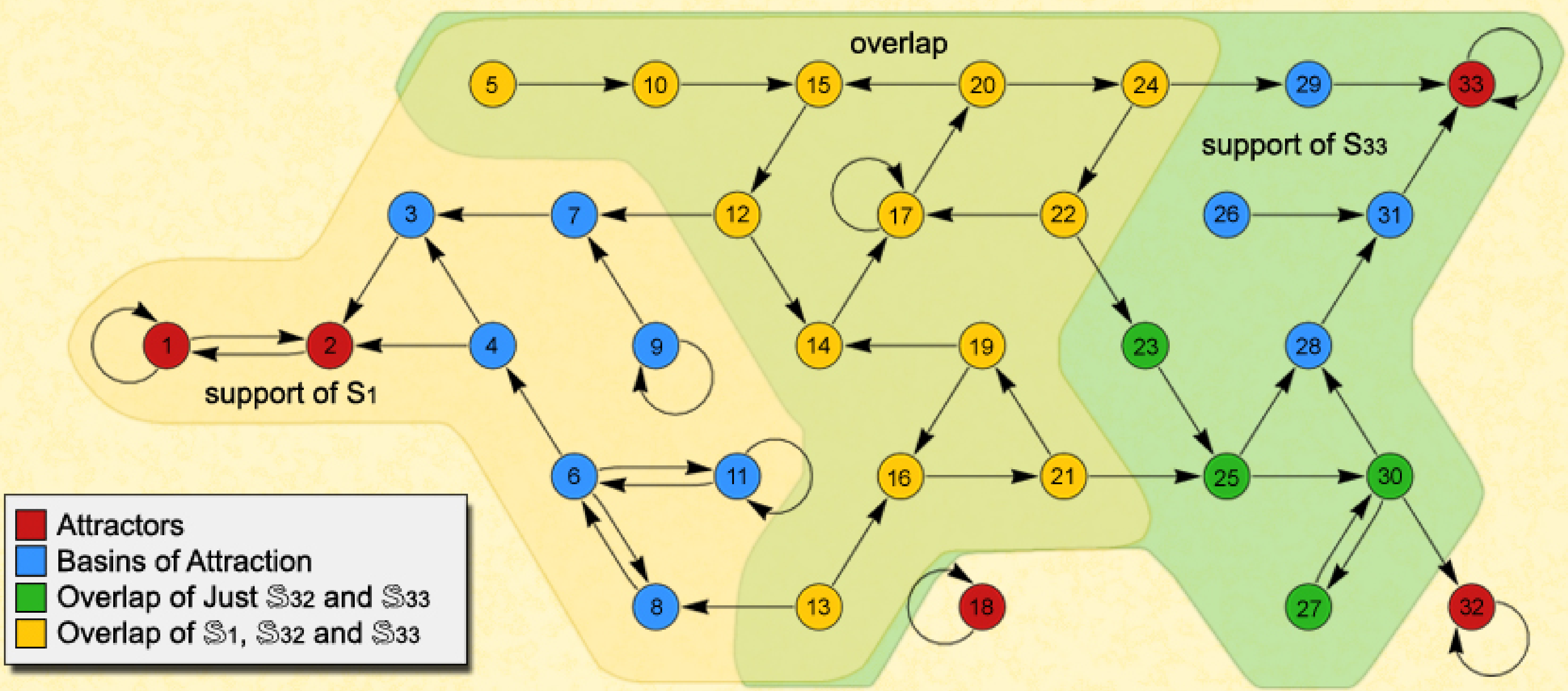


Landmarks in System Dynamics

- The overlap of a collection of states is the set of states in all of their in-components (i.e. the intersection of thier supports).

- A state's out-degree is the number of distinct successor states (states that may be immediate transitioned into). The out-degree k_i of state S_i equals $|\{S_j : P(s_{t+1} = S_j \mid s_t = S_i) > 0\}|$.

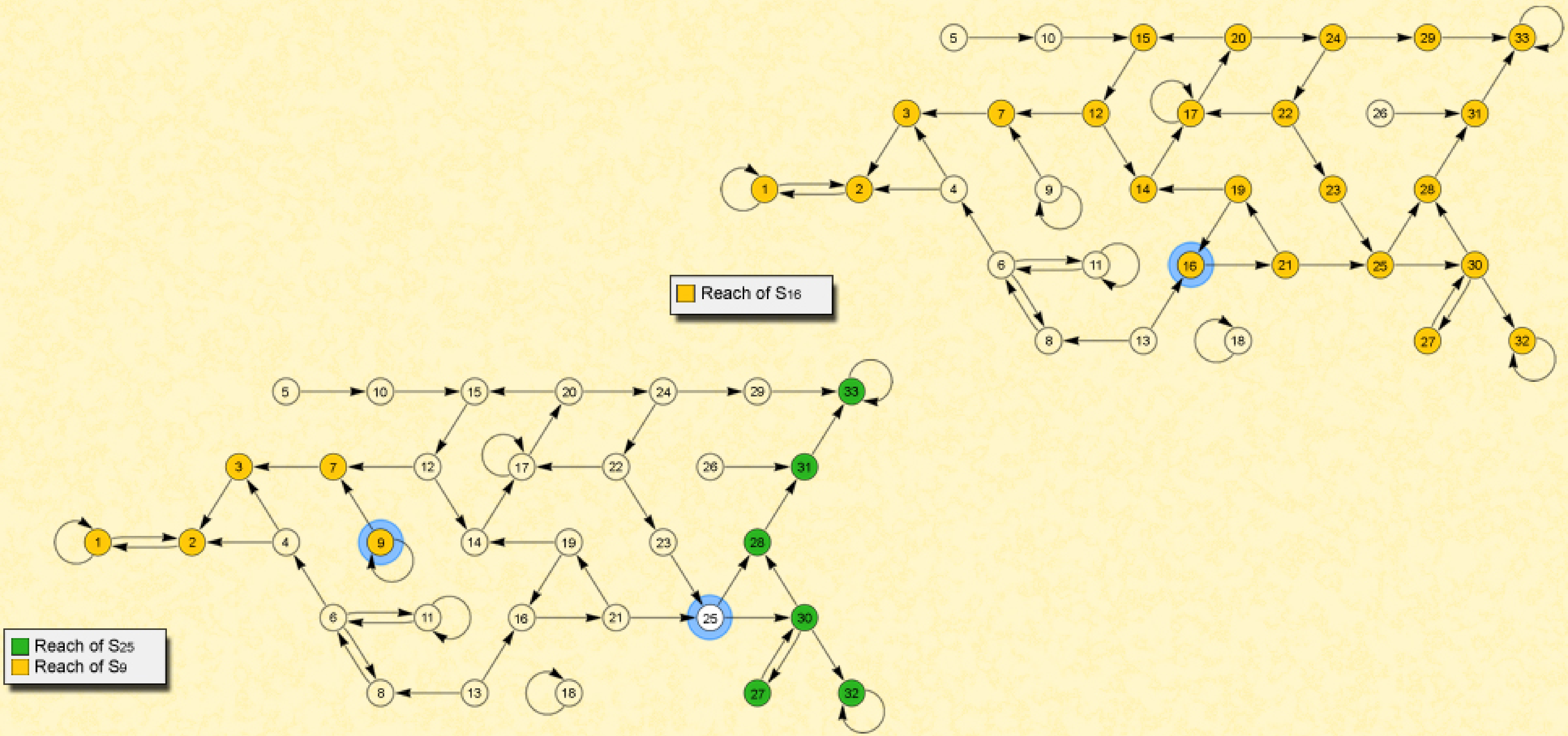
- S_k will be used to denote a neighboring state and $\mathbf{S}_k$ the set of neighboring states.



Landmarks in System Dynamics

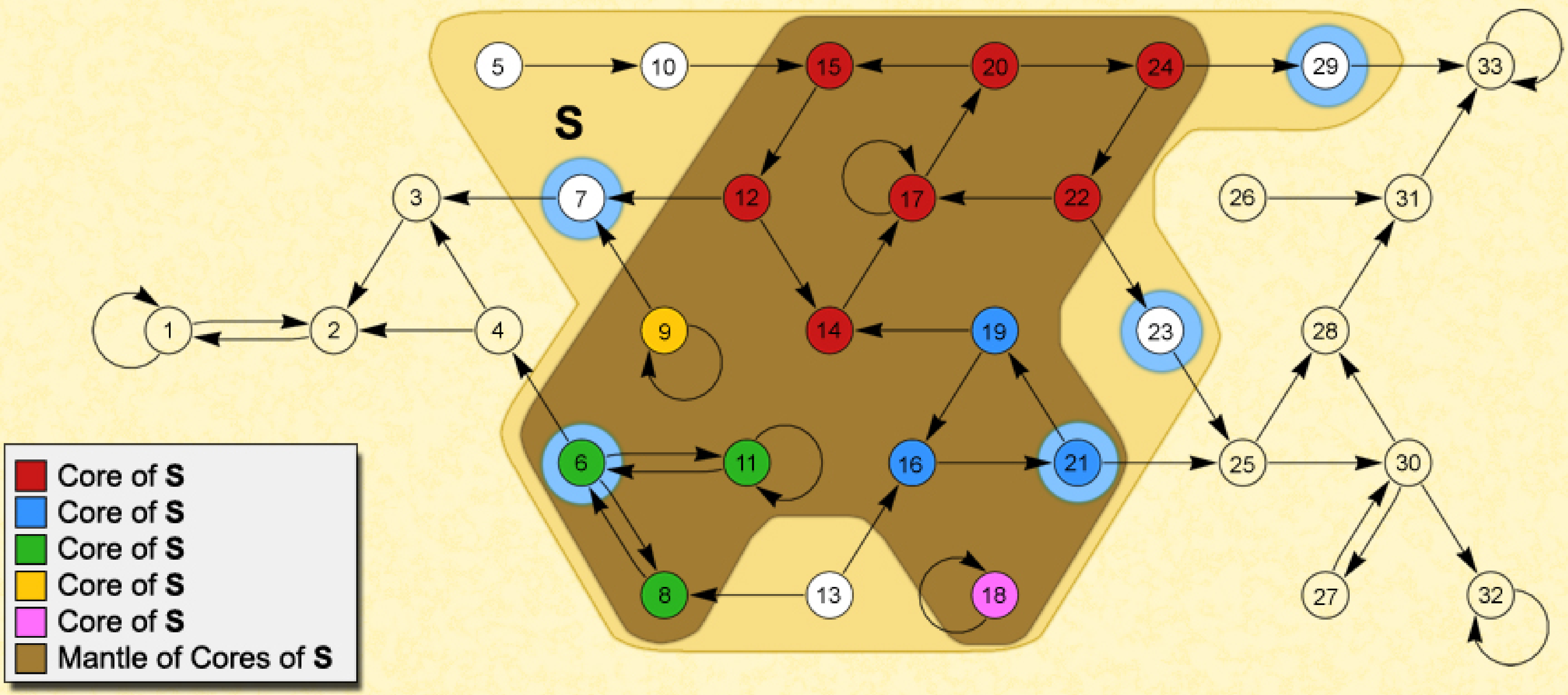
- The reach of a state (also called its out-component and written $\mathbf{R}(S_i)$) is the set of states that the system may enter by following some sequence of transitions; i.e. all possible future states given an initial state.

- Every successor state's reach is less than or equal to the previous state's reach. For every i and j, $\sim(S_i, S_j)$ implies $|\mathbf{R}(S_i)| \geq |\mathbf{R}(S_j)|$.



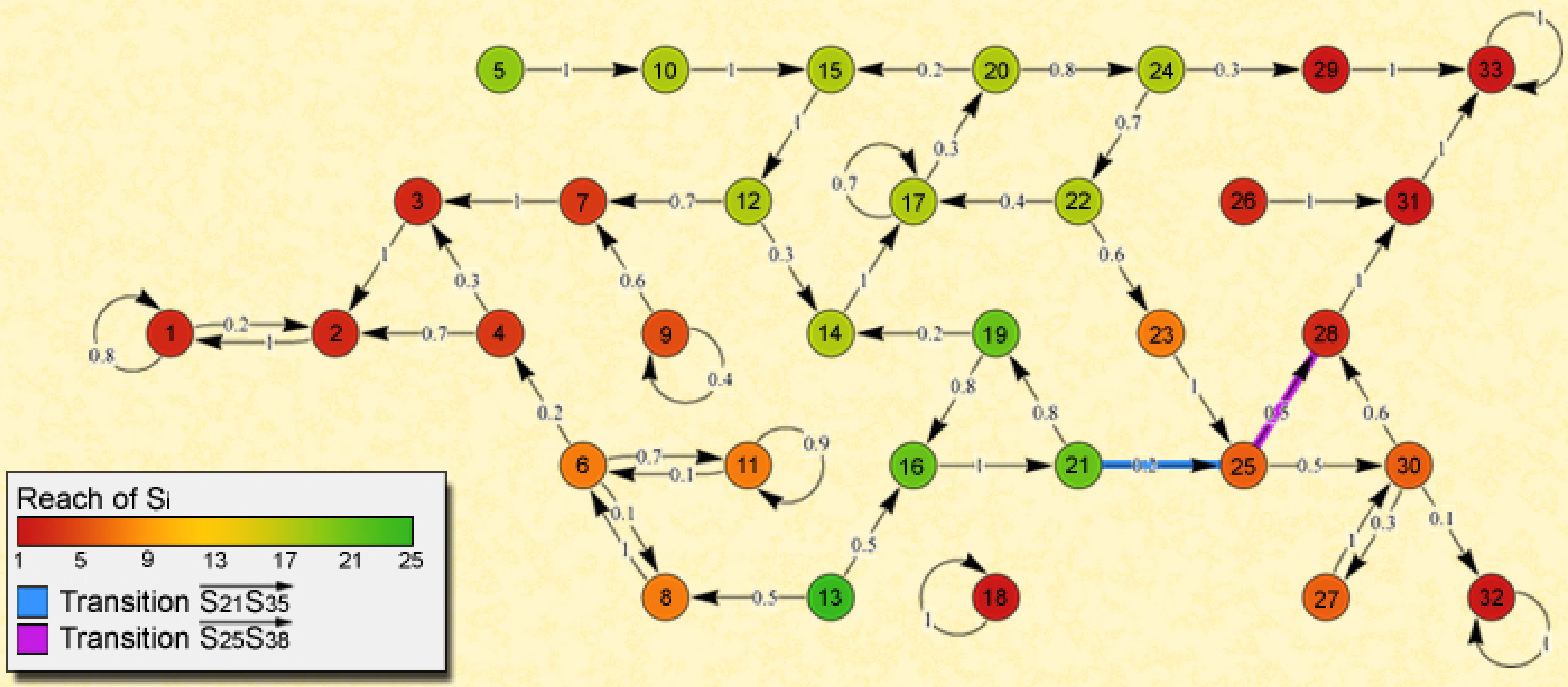
Landmarks in System Dynamics

- A core of a set is a subset wherein every member of the subset is in the reach of every member of the subset. This is the strongly connected component of a selected collection of states.
- Some sets will have multiple cores – the set of *S*'s cores can be called *S*'s mantle.
- The perimeter of a set is the collection of those states in the set that may transition to states outside the set. That is, **S** such that P(s_{t+1} not in **S** | s_t in **S**) > 0



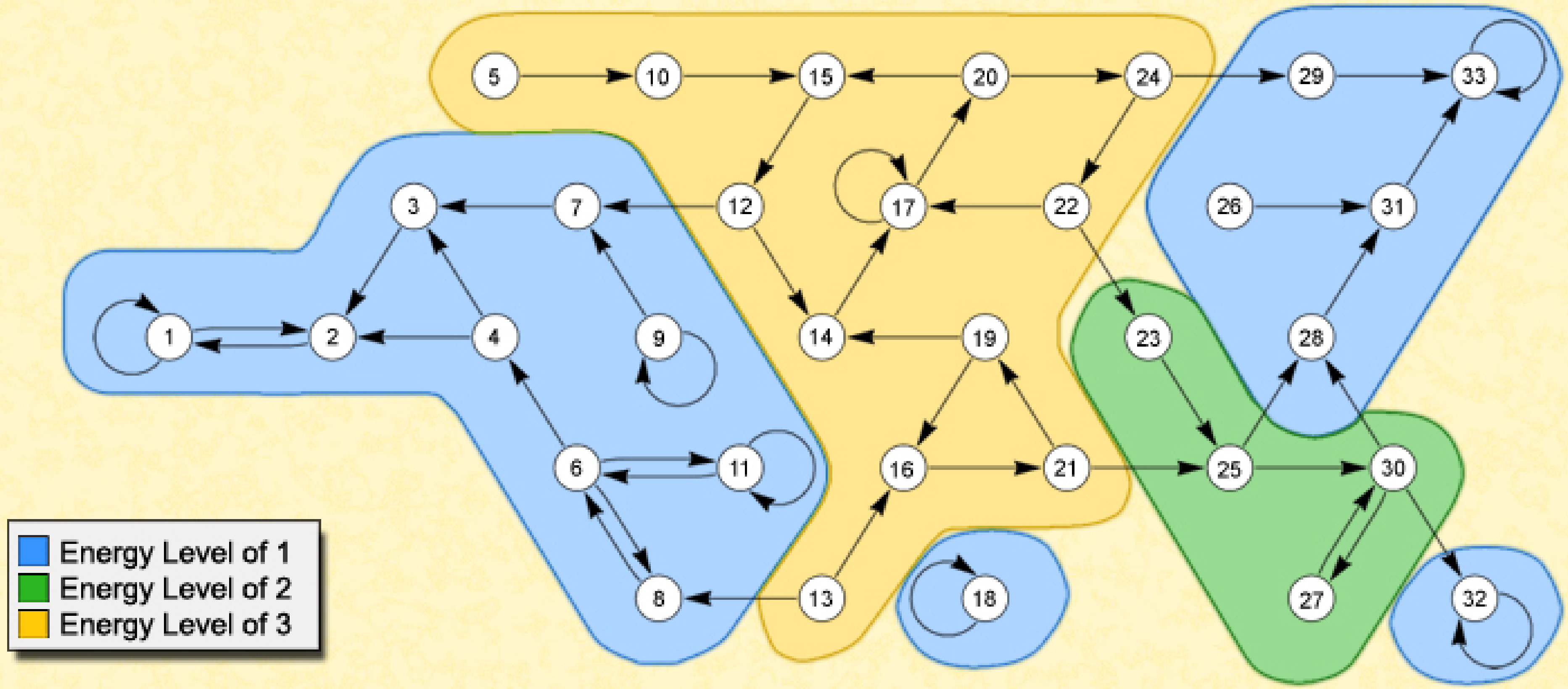
Critical Behavior

- A state transition is considered critical behavior if and only if it produces a decrease in stretch; i.e. $\sim(S_i, S_j)$ such that $|R(S_i)| > |R(S_j)|$.
- The stretch-gap of a transition is the change in the percent of the total number of states that can be reached. This quantity equals $|R(S_i)| / N - |R(S_j)| / N$.
- A **transition's** criticality is one minus the ratio of the start and end states' stretch. The criticality of $\sim(S_i, S_j)$ equals $1 - |R(Sj)| / |R(Si)|$.
- The criticality of a **state** is the probabilistically weighted sum of the criticality of all the transitions from that state.



Tipping Behavior

- A tipping point is a state which is in the perimeter of an equivalence class for some property. Different properties reveal different kinds of tips.
- The energy level of a state is the number of reference states (e.g. attractors, functional states, states with high criticality) within its reach. Energy levels partition the system's states into equivalence classes.
- The tippiness of a state S_i is the probabilistically weighted proportional drops in energy of its immediate successors: $1 - \text{Sum over } j,\ P_{ij}\ (\ E(S_j) / E(S_i)\ )$.

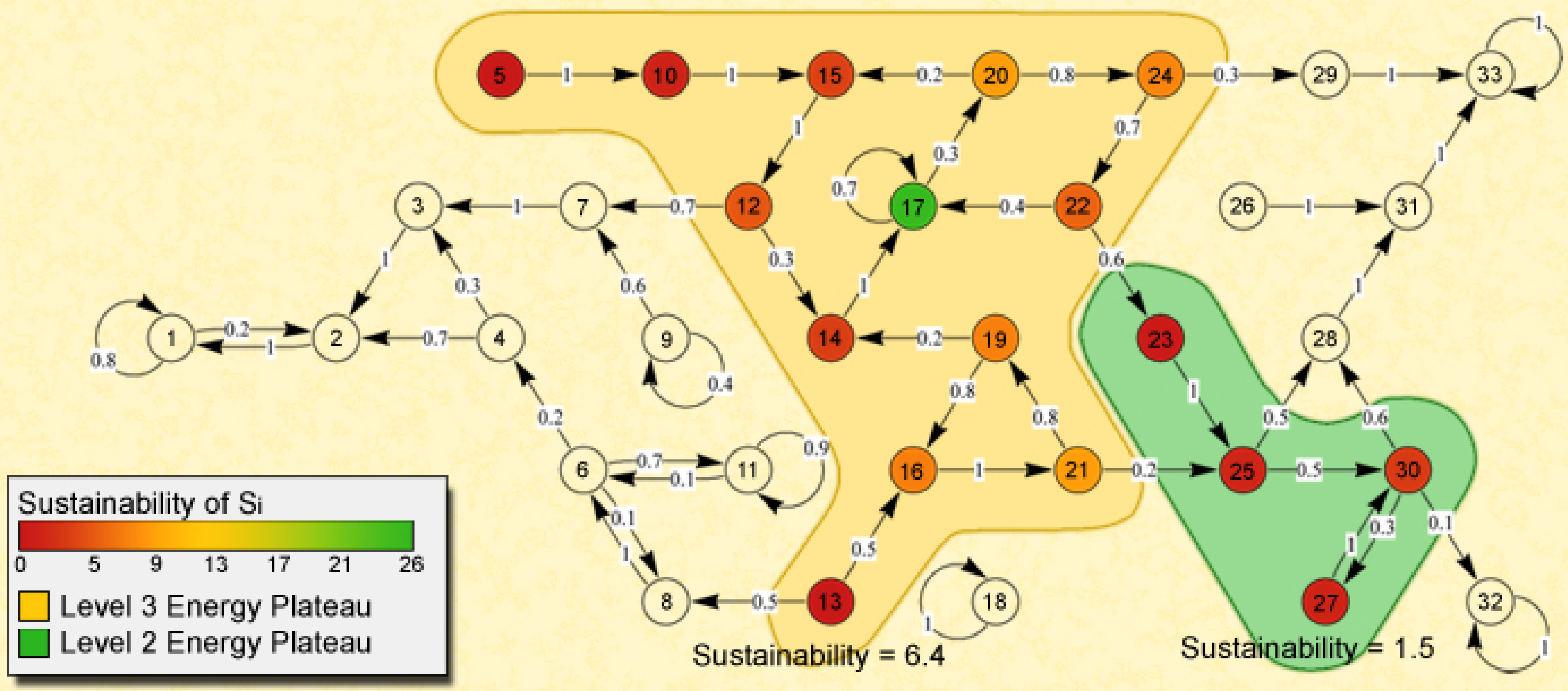


Stable, Static, and Turbulent

- A state's stability is how likely that state is to self-transition: $P(s_{t+1} = S_i | s_t = S_i)$.
- The stability of a **set** is the probability that the system will not transition out of the set given that the system starts within the set. We calculate this as the average of the individual states' **exit probabilities**.
- The degree to which a set is static is the average of the states' stability values.
- The turbulence of a set is the average percentage of states that its states can transition into. We can calculate **S**'s turbulence with the average ratio of each state's degree to the number of states in **S**: $1/|\mathbf{S}|$ Sum over S_i (k / $|\mathbf{S}|$).
- As a refinement of turbulence, weighted turbulence of the state S_i equals zero if $k = 1$ and for $k > 1$ can be calculated as Sum over j of $1 - (P(\sim(S_i, S_j)) - 1/k)^2$.

Sustainable and Susceptible

- The sustainability of **S** is the average cumulative long-term probability density of future states that remain in the set starting from each state in the set: $1/|S|$ Sum over S_i, Sum over t P(s_{t+1} in **S**|s_t in **S**).
- The degree to which **S** is susceptible to S_i is how much more (or less) likely it is to transition out of **S** conditional on it being in a particular state S_i of **S**: Sum over t P(s_{t+1} in **S**|s_t in **S** and $s_0 = S_i$) – sustainability of **S.**
- Given this definition we can see that a positive susceptibility means a lower probability to stay within **S**.



Resilient and Recoverable

- A state's resilience is the cumulative probability of returning to a state given that the system starts in that state: Sum over t $P(s_t = S_i | s_0 = S_i)$.
- **Set** resilience is the probability that the system will return to a set if the initial state of a sequence is within the state.
- A transition out of the set is recoverable to the degree that the system will return to the set after the transition. **S** is recoverable from $\sim(S_i, S_j)$ to the degree calculated by the Sum over t $P(s_t$ in **S** $| \sim(S_i, S_j)$ and S_i in **S** and S_j not in **S**$)$.
- Also note that there may be multiple paths from S_i back into **S**. Each path leading out from **S** back into **S** can be called a recovery path.

Reliable, Robust, and Vulnerable

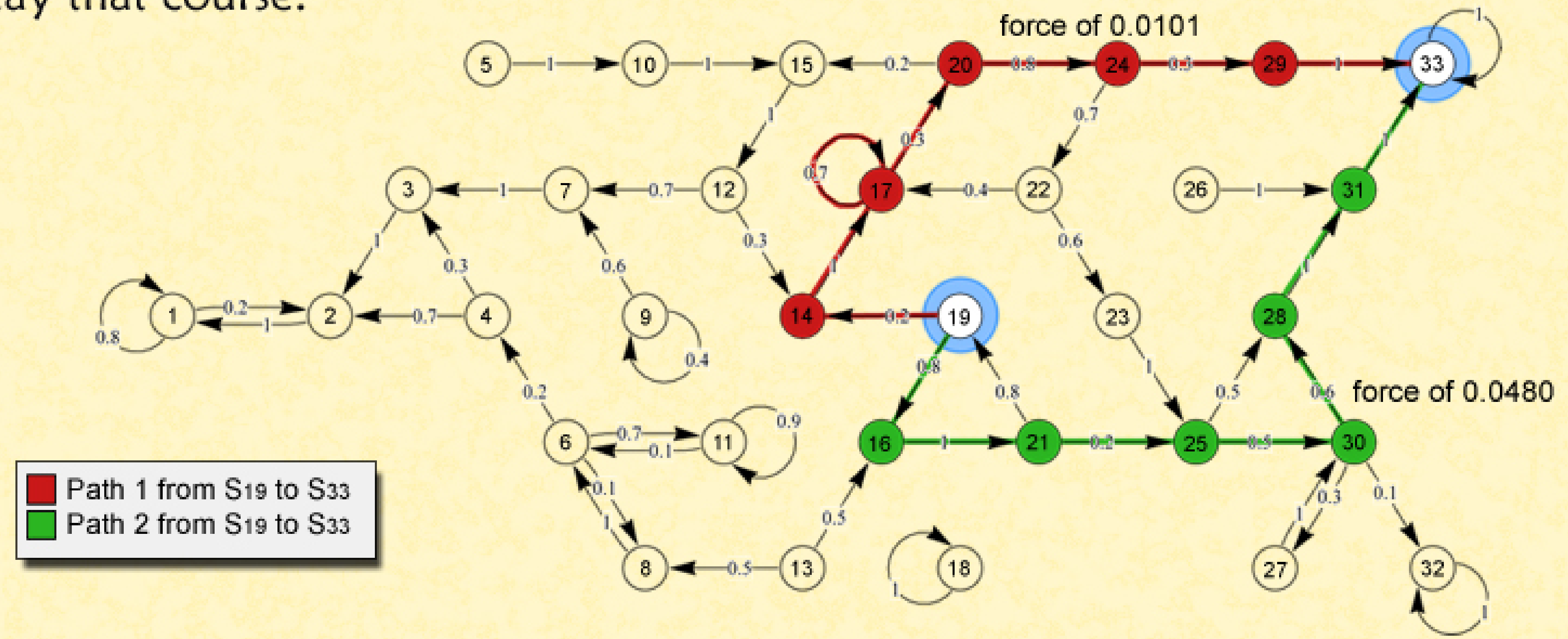
- The reliability of a set is the average cumulative long-term probability density over the states in the set given that the system starts within that set. It is by 1/|S| Sum over S_i, Sum over t P(s_t in **S**|s_0 in **S**).

- The robustness of a set is the average cumulative long-term probability density over the states in the set given that the system may start at any state. 1/|S| Sum over S_i, Sum over t P(s_t in **S**).

- A set's vulnerability at S_i is the difference in the average long-term probability density over the states in the set compared to the density generated by starting in S_i: Sum over t P(s_t in **S**|s_0 = S_i) – robustness of **S**.

Path Sensitivities

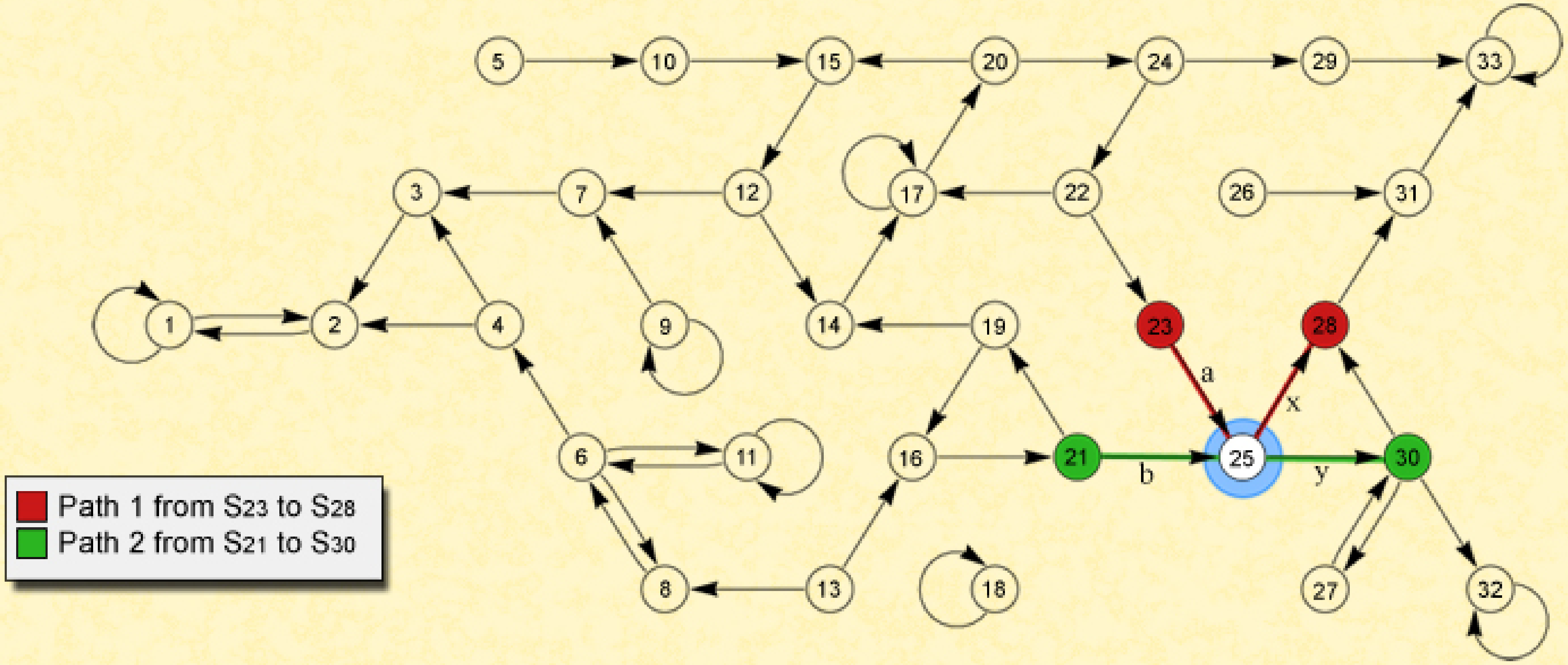
- Any reduction in the size of the reach across a transition is an instance of weak path preclusion. The degree of weak path preclusion of ~(S_i, S_j) is the criticality measure of ~(S_i, S_j).
- Strong path preclusion is when there is a reduction in the number of reference states or sets that can be reached. The strong path preclusion of ~(S_i, S_j) is measured by the tippiness of ~(S_i, S_j).
- Trajectory forcing is when a particular transition sends the dynamics down a specified sequence of states. The force of an exact path from S_i to S_j can be measured as the product of the probabilities of all the transitions required to stay that course.



Path Sensitivities

- A state's exit transitions are path dependent if and only if the distribution of their probabilities changes conditional on previous states. The degree to which S_i's transitions are path dependent on a set of historical sets S_H equals Sum over j of neighbors $(P(\sim(S_i, S_j) - P(\sim(S_i, S_j) | S_H))^2$.

- Assume that the system dynamics enter S_{25} equally often from both S_{23} and S_{21}. Given the system is at S_{25} let's assume $P(x) = P(y) = 0.5$. Analyzing the individual time series of data may reveal that $P(x|a) = 0.8$ and $P(y|b) = 1.0$.
So S_{25}'s path dependence on $S_{23} = (0.5 - 0.8)^2 + (0.5 - .2)^2 = 0.18$.
S_{25}'s path dependence on $S_{21} = (0.5 - 0.0)^2 + (0.5 - 1.0)^2 = 0.5$.



So Much More to Do

- I've created algorithms for most, but not all these measures.
- There is clearly room for more measures of the properties of system dynamics using this Markov representation.
- What about other representations? What measures can't be captured this way? Can we develop non-probabilistic measures of these phenomena?
- How much does interpretation affect the measure descriptions?
- Do application results match intuitions about these measures?
- Can we use this to find equivalence classes for system dynamics?